Methods of Energy Transport

There are 3 methods for energy transport within a star

- Radiative absorption and emission
- Convection
- Conduction (collisions in random thermal motion)

Heat Conduction is only important under degenerate conditions (*i.e.,* in white dwarfs and neutron stars). However, in normal stars, both radiation and convection can be important.

Radiative Transport of Energy

Consider a system of particles diffusing across a boundary in the z direction. Let

$n =$ the particle density
$\bar{v} =$ the mean velocity of the particles
$l =$ the particle mean free path. For particles, l is defined as the distance a particle will move before its cross-section for collision, σ, encounters another particle, $\sigma l = 1/n$

If we assume that the motions of the particles are isotropic, then on average, $\sim 1/3$ of the particles will have their motions primarily in the z direction, and $\sim 1/2$ of those will be moving toward the boundary, as opposed to away from it. Thus, the flux of particles diffusing across the boundary will be

$$F_1 = \frac{1}{6} n_{z-l}\, \bar{v}_{z-l} \tag{3.1.1}$$

Similarly, the flux of particles diffusing across the boundary from the other direction is

$$F_2 = \frac{1}{6} n_{z+l}\, \bar{v}_{z+l} \tag{3.1.2}$$

If $v_{z-l} \approx v_{z+l}$, then the net flux of particles will be

$$F_1 - F_2 = \frac{1}{6} \bar{v} \left[n_{z-l} - n_{z+l} \right]$$

If the mean free path is smaller than the density gradient, this is simply

$$F = -\frac{1}{3} \bar{v}\, l \, \frac{dn}{dz} \longrightarrow F = -D \, \nabla n \tag{3.1.3}$$

where D is the diffusion coefficient. (This is Fick's first law of diffusion.) In the case above, $D = \frac{1}{3} \bar{v}\, l$.

To compute the flux of radiative energy across a boundary, treat the photons as particles, and recall that the energy density of radiation, U, is

$$U = a\,T^4$$

Thus

$$F = -\frac{1}{3}\bar{v}\,l_{\rm ph}\,\frac{dU}{dr} = -\frac{1}{3}\,c\,l_{\rm ph}\left(4\,a\,T^3\,\frac{dT}{dr}\right)$$

For photons, the mean free path between collisions is

$$l_{\rm ph} = \frac{1}{\kappa\rho}$$

where κ is the mean absorption coefficient (with units of area per unit mass), so the diffusion equation can be written

$$F = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr} \tag{3.1.4}$$

To put this in terms of an expression useful for stellar structure calculations, first substitute $\mathcal{L}$ for F,

$$\frac{\mathcal{L}}{4\pi r^2} = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

and then change the equation to the Lagrangian form

$$\frac{dT}{d\mathcal{M}} = -\frac{3\kappa\mathcal{L}}{64\pi^2 ac\,r^4T^3} \tag{3.1.5}$$

We can now put this in terms of the radiative temperature gradient for a star in (at least approximate) hydrostatic equilibrium

$$\nabla_{\rm rad} = \left(\frac{d\ln T}{d\ln P}\right)_{\rm rad} = \frac{\mathcal{M}}{T}\frac{dT}{d\mathcal{M}}\bigg/\frac{\mathcal{M}}{P}\frac{dP}{d\mathcal{M}}$$

Substituting the Lagrangian equation for the radiative temperature gradient (3.1.5) and the pressure (2.2.4) we get

$$\nabla_{\rm rad} = \frac{3\kappa \mathcal{L} P}{16\pi a c G \mathcal{M} T^4} \tag{3.1.6}$$

and the radiative plus conductive flux at any point in the star is

$$F_{\rm rad} = \frac{4\, a\, c\, G \mathcal{M} T^4}{3\kappa\, r^2\, P}\, \nabla_{\rm rad} \tag{3.1.7}$$

Note that the diffusion approximation is only good when the photon mean free path is short compared to the scale of the density gradient. Near the surface, the approximation breaks down.

Conductive Transport of Energy

Energy transport by conduction (*i.e.*, from collisions between electrons and atomic nuclei) works exactly the same as radiative transport – through Fick's equation. Thus,

$$F = \frac{dq}{dt} = -D_e \frac{dT}{dr}$$

and the conductive term can simply be combined with the radiation term. If we define "conductive opacity" as

$$\kappa_{\rm cd} = \frac{4acT^3}{3D_e\rho} \tag{3.1.8}$$

then

$$F_{\rm cd} = \frac{4acT^3}{3\kappa_{\rm cd}\rho}\frac{dT}{dr}$$

and

$$\frac{1}{\kappa_{\rm tot}} = \frac{1}{\kappa_{\rm rad}} + \frac{1}{\kappa_{\rm cd}} \tag{3.1.9}$$

Criteria for Convection

To decide whether convection occurs in a star, begin by considering a blob of material with volume V in pressure equilibrium with its surroundings. Initially, both the blob and its surroundings have temperature T_1 and density ρ_1. Now suppose the blob's temperature is perturbed, such that its new temperature is T_1'. If $T_1' > T_1$ and pressure balance is maintained, then the equation of state of the gas demands that $\rho' < \rho$. Given a local gravity, g, the blob will feel a buoyance force per unit volume

$$\rho_1 g - \rho_1' g$$

As a result, the blob will rise a distance Δr, to a location in the star with T_2 and ρ_2. As it does so, the blob will expand to a larger volume (to maintain pressure equilibrium with its surroundings) and its new density will be ρ_2'. The buoyancy force on the blob will now be

$$\rho_2 g - \rho_2' g$$

If $\rho_2' < \rho_2$, then the blob will continue to rise, and the star will be dynamically unstable to convection; if $\rho_2' > \rho_2$ then the blob will sink back to its original position. In other words, convection will *not* occur if

$$\left(\frac{d\rho}{dr}\right)_i > \left(\frac{d\rho}{dr}\right)_s \tag{3.2.1}$$

where the subscript i denotes how the internal density of the blob has changed with radius, and the subscript s indicates the gradient in the star.

To translate this condition for stability into a more tractable form, begin by using the equation of state, $\rho = \rho(P, T, \mu)$. $d\rho$ is then related to P, T, and μ by

$$d\rho = \left(\frac{\partial \rho}{\partial P}\right) dP + \left(\frac{\partial \rho}{\partial T}\right) dT + \left(\frac{\partial \rho}{\partial \mu}\right) d\mu$$

so

$$
\begin{aligned}
\frac{d\rho}{\rho} &= \frac{P}{\rho}\left(\frac{\partial\rho}{\partial P}\right)\frac{dP}{P} + \frac{T}{\rho}\left(\frac{\partial\rho}{\partial T}\right)\frac{dT}{T} + \frac{\mu}{\rho}\left(\frac{\partial\rho}{\partial \mu}\right)\frac{d\mu}{\mu} \\
&= \left(\frac{\partial\ln\rho}{\partial\ln P}\right)\frac{dP}{P} + \left(\frac{\partial\ln\rho}{\partial\ln T}\right)\frac{dT}{T} + \left(\frac{\partial\ln\rho}{\partial\ln\mu}\right)\frac{d\mu}{\mu} \\
&= \alpha\,\frac{dP}{P} - \delta\,\frac{dT}{T} + \varphi\,\frac{d\mu}{\mu} \qquad (3.2.2)
\end{aligned}
$$

where α is the isothermal compressibility, δ is the volume coefficient of expansion, and φ is the chemical potential coefficient. Thus, the condition for stability is then

$$\left(\frac{\alpha}{P}\frac{dP}{dr} - \frac{\delta}{T}\frac{dT}{dr} + \frac{\varphi}{\mu}\frac{d\mu}{dr}\right)_i > \left(\frac{\alpha}{P}\frac{dP}{dr} - \frac{\delta}{T}\frac{dT}{dr} + \frac{\varphi}{\mu}\frac{d\mu}{dr}\right)_s$$

Since, by definition, the blob must remain in pressure equilibrium with its surroundings, the first terms on either side of the inequality cancel. Furthermore, since the blob does not change its chemical composition while moving, $(d\mu/dr)_i = 0$. This leaves

$$\left(\frac{\delta}{T}\frac{dT}{dr}\right)_i < \left(\frac{\delta}{T}\frac{dT}{dr} - \frac{\varphi}{\mu}\frac{d\mu}{dr}\right)_s$$

Now if we multiply both sides of the inequality by the pressure scale height,

$$\lambda_P = -P\Big/\frac{dP}{dr} = -\frac{dr}{d\ln P} \qquad (3.2.3)$$

the condition for stability becomes

$$\left(\delta\,\frac{d\ln T}{d\ln P}\right)_i > \left(\delta\,\frac{d\ln T}{d\ln P} - \varphi\,\frac{d\ln\mu}{d\ln P}\right)_s$$

or

$$\left(\frac{d\ln T}{d\ln P}\right)_s < \left(\frac{d\ln T}{d\ln P}\right)_i + \frac{\varphi}{\delta}\left(\frac{d\ln \mu}{d\ln P}\right)_s \tag{3.2.4}$$

Note that if the star is stable, the term on the left is $\nabla_{\rm rad}$, the star's radiative temperature gradient. Moreover, the first term on the right is nothing more than the adiabatic temperature gradient. Thus, for a chemically homogeneous region of a star, the condition for stability is simply

$$\nabla_{\rm rad} < \nabla_{\rm ad} \tag{3.2.5}$$

This is the Schwartzschild criterion for dynamical stability. If the star is chemically inhomogeneous, then the Ledoux stability criterion must be used:

$$\nabla_{\rm rad} < \nabla_{\rm ad} + \frac{\varphi}{\delta}\left(\frac{d\ln \mu}{d\ln P}\right) \tag{3.2.6}$$

The Schwartzschild criterion can be translated into a statement about entropy. In effect, the criterion is equivalent to saying that the specific entropy of a star never decreases as you move outward in radius. To see this, consider the thermodynamic relation

$$dq = T ds = c_P T\left[\frac{dT}{T} - \nabla_{\rm ad}\frac{dP}{P}\right] \tag{1.21}$$

If we take the derivative of both sides with respect to r, and change to log quantities, then the radial gradient of entropy is given by

$$\frac{ds}{dr} = c_P\left\{\frac{d\ln T}{dr} - \nabla_{\rm ad}\frac{d\ln P}{dr}\right\} \tag{3.2.7}$$

So, if we take the pressure derivative out of the brackets, we get

$$\frac{ds}{dr} = c_P\left\{\frac{d\ln T}{d\ln P} - \nabla_{\rm ad}\right\}\frac{d\ln P}{dr} \tag{3.2.8}$$

or

$$\frac{ds}{dr} = c_P \left(\nabla - \nabla_{\rm ad} \right) \frac{d \ln P}{dr} \tag{3.2.9}$$

Obviously, for a star to be in hydrostatic equilibrium, the pressure must decrease outward, so $d \ln P/dr$ is negative. Therefore, if the star is dynamically stable, $\nabla < \nabla_{\rm ad}$ and the star's entropy derivative is positive. If the star is unstable to convection, then $\nabla = \nabla_{\rm ad}$ and the entropy will be constant over the region. (Actually, it is possible for ∇ to exceed $\nabla_{\rm ad}$ by a very small amount, but we'll get to this later.)

Convective Transport of Energy

In a radiative region of a star, the temperature gradient is given by ∇_{rad}. However, energy transport by radiation (and conduction) is not always efficient. In that case, convection transport of energy will occur. By definition, if radiation and conduction are not working, then energy is not being lost from the individual convective cells, and the stellar temperature gradient $\nabla = \nabla_{\mathrm{ad}}$. If energy transport by radiation and convection are comparable then $\nabla_{\mathrm{rad}} > \nabla > \nabla_{\mathrm{ad}}$.

In practice, when convection occurs in stellar cores, $\nabla \sim \nabla_{\mathrm{ad}}$. Only in the outer parts of some stars do *superadiabatic* conditions exist.